Poisson

Geometric

Memoryless property.

4.5 Exponential distribution

Recall $X \sim \text{Geom}(p) \longrightarrow \{1, 2, \ldots\}$ (discrete random variable)

= "Waiting time until first success".

But $X = 1, 2, 3, \ldots$ etc so it is is discrete.

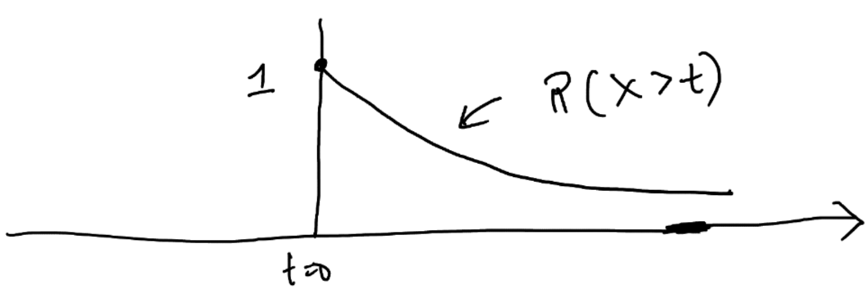
$X \sim \text{Exp}(\lambda)$ is a waiting time that is continuous.

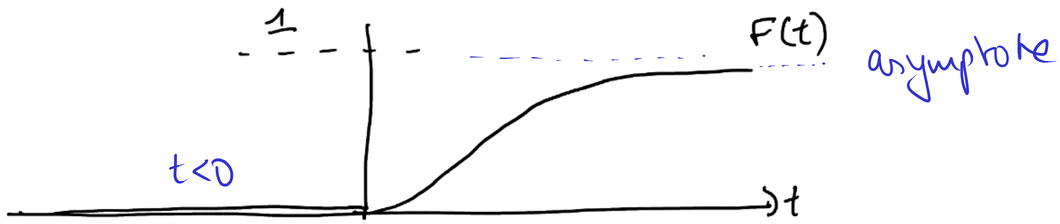
cdf

$F(t) = P(X \le t)$

$\boxed{P(X > t) = e^{-\lambda t}}$ (Tail) $t \ge 0$. λ is some parameter like 4.

$= 1 - F(t) \iff F(t) = 1 - e^{-\lambda t} \quad t \ge 0$





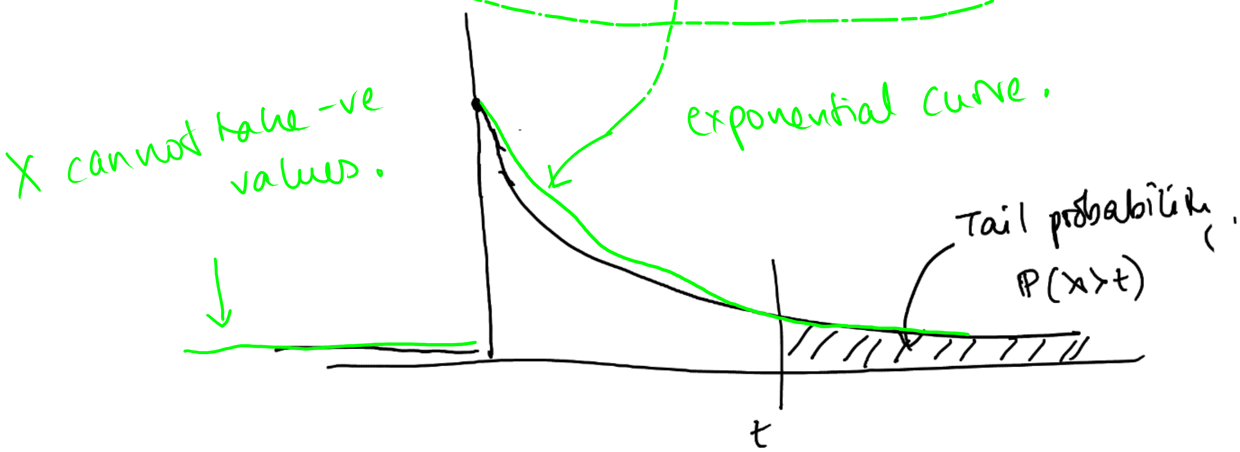
The pdf is the derivative of the cdf.

$$F(t) = 1 - e^{-\lambda t} \qquad t \geq 0$$

Therefore: $f(t) = \frac{d}{dt} F(t)$. (use the chainrule)

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

write down in your cheatsheet



X models a waiting time $\Rightarrow$ $X \geq 0$

Expectation and variance

differentiate this away

(When in doubt)
↳ you integrate by parts.

$$E[X] = \int_0^\infty \underbrace{t}_{\text{(value)}} \underbrace{\lambda e^{-\lambda t}}_{\text{(probability)}} dt$$

↳ pdf.

$$u = t \quad dv = \lambda e^{-\lambda t} dt \Rightarrow v = (-e^{-\lambda t})$$

$$= t\,(\cancel{-e^{-\lambda t}})\Big|_0^\infty - \int_0^\infty 1 \cdot (-e^{-\lambda t})\, dt = \int_0^\infty e^{-\lambda t} dt = \frac{e^{-\lambda t}}{-\lambda}\Bigg|_0^\infty$$

$$= \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

(we will compute this more efficiently using mgfs later)

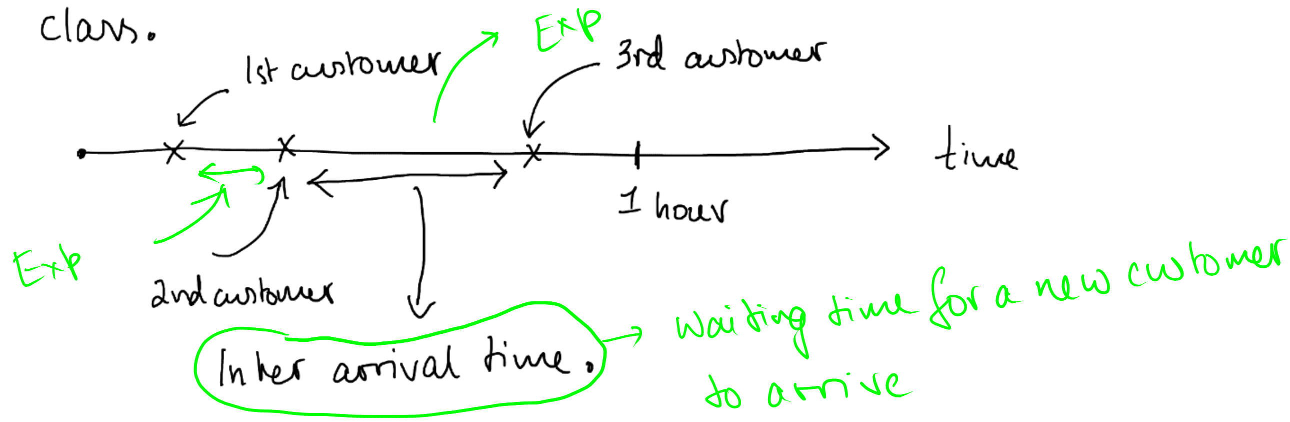
↓

You have to integrate by parts twice.

Apply HW.

Exponential and Poisson

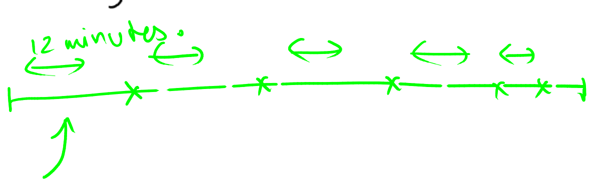
We will not cover this in detail in this class.



This interarrival time is $Exp(\lambda)$.

The inter arrival times are also independent.

If $\lambda = 5$ customers/hour. Then the average time between customers coming in should be $\frac{1}{5}$ hour, or every 12 minutes.



$$\frac{60}{12} = 5$$

$Y \sim Exp(\lambda)$ $\quad E[Y] = \frac{1}{\lambda} = \frac{1}{5}$ (as required)

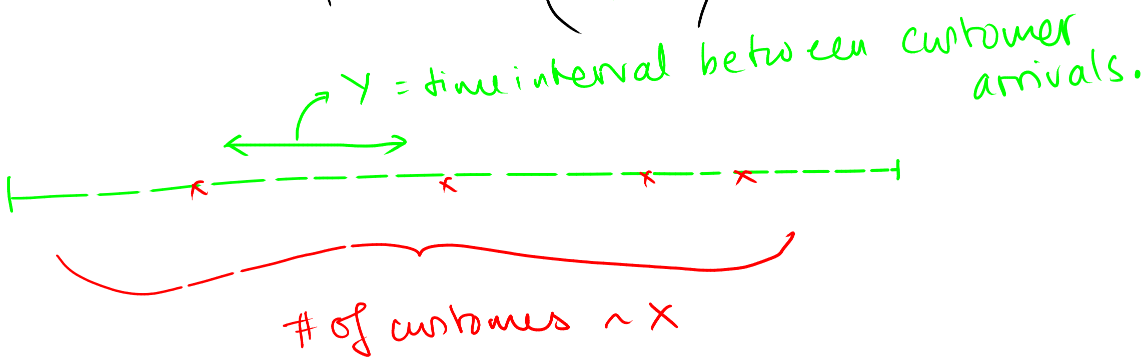
Let

X = # of customers that come to the store on one day (say Jan 7)

Assume $X \sim \text{Poisson}(\lambda)$

Let Y = Time between successive customer arrivals.

Then $Y \sim \text{Exponential}(\lambda)$



Memory less property

$X \sim$ waiting time $\sim Exp(\lambda)$

Suppose you have waited 3 minutes already. What is the probability that you wait 5 more minutes?

Let's unpack that. X = waiting time.

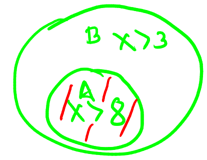
Waited 3 minutes and it has not already happened : $X > 3$ given

Probability of having to wait at least 5 more minutes : $X > 5+3$.

This is what you're asked to compute. $\leftarrow \mathbb{P}(X > 5+3 \mid X > 3)$

$$= \frac{\mathbb{P}(X > 5+3 \cap X > 3)}{\mathbb{P}(X>3)}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



$$= \frac{P(X > 8)}{P(X > 3)}$$

$$P(X>t) = e^{-\lambda t}$$

$$P(X>5+3 \mid X>3) = \frac{e^{-\lambda 8}}{e^{-\lambda 3}} = e^{-5\lambda} = P(X > 5)$$

Geometric and Exponential

Suppose

$$X \sim \text{Exp}(\lambda)$$

Then $$P(X > t) = e^{-\lambda t}$$

Suppose $$Y \sim \text{Geometric}(p)$$

Then $$P(Y > t) = P(\text{1st } t \text{ trials are failures})$$

$$= (1-p)(1-p) \cdots \underbrace{(1-p)}_{t^{th} \text{ trial}}$$

$$= (1-p)^t = e^{t \underbrace{\log(1-p)}}$$

$$\lambda = -\log(1-p)$$

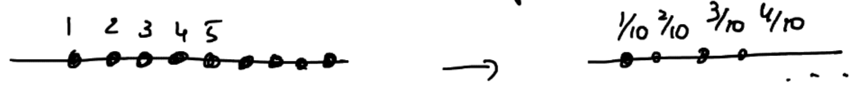
Geometrics appropriately scaled $\Rightarrow$ Exponential.

Recall p = probability of success.

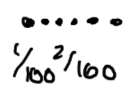
INTERESTING | UNSURPRISING FACT:

The exponential is a limit of a geometric random variable!

I won't make this precise, but if



(squeeze points together)



then this goes to the continuous random variable $X \sim Exp(\lambda)$.

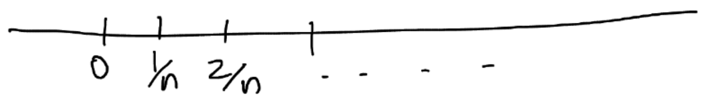
See theorem 4.33 in your textbook for details.

In case you're really curious, here is a short proof of this fact.

The Geometric distribution converges to exponential

I usually skip this: SAFELY IGNORE.

$$\frac{1}{n}\,\mathrm{Geom}(p) = \frac{Y_n}{n}$$



$$\mathbb{P}\left(\frac{\mathrm{Geom}(p)}{n} > x\right) = \sum_{k=\lceil nx \rceil}^{\infty} \mathbb{P}(Y_n = k)$$

$$= \sum_{k=\lceil nx \rceil}^{\infty} (1-p)^{k-1}\, p$$

$$= (1-p)^{\lceil nx \rceil - 1}\, \cancel{p}\, \frac{1}{\cancel{1-1-p}}$$

let $p = \frac{\lambda}{n}$.

using the geometric formula.

$$= \left(1 - \frac{\lambda}{n}\right)^{\lceil nx \rceil - 1} \approx \frac{\left(1 - \frac{\lambda}{n}\right)^{nx}}{\left(1 - \frac{\lambda}{n}\right)} = \frac{\left(\left(1 - \frac{\lambda}{n}\right)^{n}\right)^{x}}{1 - \frac{\lambda}{n}}$$

$$\longrightarrow \frac{\left(e^{-\lambda}\right)^{x}}{1} = \mathbb{P}(\mathrm{Exp}(\lambda) > x)$$